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INTERNATIONAL CONFERENCE of SCIENTIFIC PAPER
AFASES 2011
Brasov, 26-28 May 2011

NUMERICAL APPROXIMATION OF DIRICHLET PROBLEM IN BOUNDED DOMAINS AND APPLICATIONS

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Abstract: *We consider numerical approximation of Dirichlet problem for the Laplace equation in a domain* $D \in R^d$, *that is we will consider the problem of finding a* C^2 *function* $u = u(z) \in C^2(D) \cap C^0(\overline{D})$ *such that* $\begin{cases} \Delta u = 0, in D \\ u = f, on \partial D \end{cases}$. *Using probabilistic methods we can give explicit reprezentation of solution of Dirichlet problem* $u(z) = E^z f(B_{\tau_D})$, *where* B_t *is a Brownian motion starting at* $B_0 = z$, E^z *denotes the expectation of function in* B_{τ_D}, *and* $\tau_D = \inf\{t \geq 0, B_t \notin D\}$ *is the exit time of Brownian motion from D. We give a Mathematical implementation of function* $u(z)$ *for different choices of* f *and domain D (half-plane, unit disc, rectangle, triangle) and we apply it to obtain some numerical results.*

Mathematics Subject Classifications 2011: *60J65, 35J05, 65C30, 68U20.*
Keywords: *Brownian motion, Dirichlet problem, Laplace equation, stochastic differential equations, simulation.*

1. INTRODUCTION

The Dirichlet Problem is named after German mathematician Gustav Lejeune Dirichlet (1805- 1859) (see [3]).

The Dirichlet problem for harmonic functions always has a solution, and that solution is unique, when the boundary is sufficiently smooth and *f* is continuous.

The goal of the present paper is to present some applications of Brownian motion in solving classical differential equations: the Dirichlet problem.

Brownian motion, named after the Scottish botanist Robert Brown in 1828, is the unique process with the following proprieties:

a) No memory, which means that $B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, B_{t_3} - B_{t_2}, \ldots$ are independent;
b) Invariance, which means that the distribution of $B_{s+t} - B_s$ depends only on t;
c) Continuity which means that $t \to B_t$ is continuous a.s. and $t \to B_t$ is nowhere differentiable a.s.
d) $B_0 = 0$, with mean $E(B_t) = 0$ and variance $Var(B_t) = t^2$.

Definition. A *d*-dimenional Brownian motion starting at $x \in R^d$ is a stochastic process B_t with the following proprieties:

a) $B_0 = 0$;

b) For all $0 \le s < t$, $B_t - B_s$ is a normal random variable $N(0, t-s)$;

c) B_t is almost surly continuous.

2. THE DIRICHLET PROBLEM

2.1 The Dirichlet Problem. We will consider the well-known Dirichlet Problem for a domain $D \subset R^d$, this is we will consider the problem of finding a harmonic function in a given domain D, continuous on $\overline{D}$, with fixed boundary values on ∂D, satisfying the following initial value problem: find $u \in C^2(D) \cap C^0(\overline{D})$ which solves

$$\begin{cases} \Delta u = 0 \text{ in D} \\ u|_{\partial D}(x,y) = f(x,y), \forall z = x + iy \in \partial D \end{cases} \quad (1)$$

where Δ denote the Laplacian operator, namely the differential operator in the variable $x = (x_1, x_2, ..., x_n \in R^d)$

$$\Delta_x = \sum_{j=1}^{d} \left(\frac{\partial}{\partial x_j} \right)^2$$

and f is a given function, continuous on boundary of the domain D.

In general, the solution of the above boundary value problem may not exist. However the existence of the solution is closely related to the regularity of the boundary of the domain D.

Definition. A point $x \in R^d$ is called regular for the set $A \subset R^d$ if a Brownian motion starting at x enters the set A immediately, that is

$$P^x(T_A = 0) = 1,$$

where $T_A = \inf\{t > 0 : B_t \in A\}$ is the hitting time of the set A by a *d*-dimensional Brownian motion B_t starting at x.

Example. a) The point $z_0 = 0$ is regular for the ball $B(1,1)$, but is not regular for $B(0,1) \setminus \{0\}$.

b) In the case of unit disk $U = \{x \in R^2 : |x| < 1\}$, all points on the unit circle $\partial U = \{x \in R^2 : |x| = 1\}$ are regular for U^c.

Under minimal regularity conditions on D and f, the main result is the following:

Theorem. Let $D \subset R^d$ be a bounded domain for which every point of ∂D is regular for D^c. If $f : \partial D \to R$ is a continuous function, then there exists a unique solution of the Dirichlet problem (1), explicitly given by

$$u(x) = E^x f(B_{\tau_D}), \quad (2)$$

where

- B_t is a *d*-dimensional Brownian motion starting at $x \in \overline{D}$;
- $\tau_D = \inf\{t > 0 : B_t \notin D\}$ is the lifetime of the Brownian motion B_t, killed on exiting D;
- E^x denotes the expectation of function f in the exit point B_{τ_D} of the Brownian motion B_t from domain D.

Proof. See [4, p. 111-113].

Example. Consider the domain $D = B(0,r)$ and the function $f(x,y) = x^2 - y^2$. Then the probabilistic solution is the following:

$$u(x) = E^x f(B_{\tau_{B(0,r)}}),$$

and $u(0) = E^0 f(B_{\tau_D}) = \frac{1}{2\pi \text{ r}} \int_{\partial \text{B}(0,\text{r})} f(y) dy$.

For $\partial B(0,r)$ we have $z = re^{it}$ and $dy = rdt$.

$$\Rightarrow u(0) = \frac{1}{2\pi \text{ r}} \int_0^{2\pi} f(r\cos t, r\sin t) r dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} (r\cos t)^2 - (r\sin t)^2 dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} r^2 (\cos^2 t - \sin^2 t) dt$$

$$= \frac{r^2}{2\pi} \int_0^{2\pi} (\cos 2t) dt$$

$$= \frac{r^2}{2\pi \mathrm{r}}\left(\frac{\sin 2t}{2}\right)\Big|_0^{2\pi} = 0\,.$$

2.2 A numerical algorithm. We will consider D to be the closed triangular domain with vertices (-a, 0), (b, 0) and (0, c). We will try to elaborate an algorithm for discretizing the killed Brownian motion in a simply bounded domain, using some recent results (see [1]) and [2]).

First, if $\left(X^k_{2^{-2k}n}\right)_{n\in N}$ is a simple random walk on the lattice $D \cap 2^{-k}Z^2 = D_k$ which jumps to one of its nearest neighbors every 2^{-2k} units of time, we obtain that $X_t^k \xrightarrow{k\to\infty} B_t$, $t \ge 0$, for a chosen level of discretisation $k \in N$.

We consider $n = [t2^{2k}]$ and $X_0 = 0$.

The numerical approximation of the value $f(B_t)$, where B_t is a killed Brownian motion in D starting at the point $x = (x_1, x2) \in D$, is given by

$$f(B_t) \approx f(X_t).$$

Then the numerical approximation of expected value $E^x f(B_t)$ is given by

$$E^x f(B_t) = \frac{f\left(X_t^1\right) + \ldots + f\left(X_t^N\right)}{N}.$$

2.3 Using Mathematica softwere. Using Mathematica (see [5]) source presented below, we obtain the approximating domain D_1 in the Fig. 1 below, in the case of a triangle with vertices at (-4, 0), (2, 0) and (0, 6) (See article).

For an arbitrarily fixed $k \in N^*$, note that $\left(\frac{i}{2^k}, \frac{j}{2^k}\right) \in D_k$ if and only if $i, j \ge 0$ and

$$\begin{cases} -\dfrac{i}{a2^k} + \dfrac{j}{c2^k} - 1 \le 0 \\ \dfrac{i}{b2^k} + \dfrac{j}{c2^k} - 1 \le 0 \end{cases}$$

Which shows that D_k can be written as follows

$$D_k = \bigcup_{j=0}^{[c2^k]} \left\{ \left(\frac{i}{2^k}, \frac{j}{2^k}\right) : -\left[\frac{a}{c}\left(c2^k - j\right)\right] \le i \le \left[\frac{b}{c}\left(c2^k - j\right)\right] \right\}$$

```
a=4; b=2; c=6; k=3;
abc={{b,0},{0,c},{-a,0},{b,0}};
triangle={Thickness[.01],RGBColor[1,0,0],Line[abc]};
incr=(1/2^k);
x=Table[i*incr,{i,-IntegerPart[a*2^k],
IntegerPart[b*2^k]}];
y=Table[j*incr,{j,0,IntegerPart[c*2^k]}];
imin=Table[-IntegerPart[a*2^k-a*j/c],
{j,0,IntegerPart[c*2^k]}];
imax=Table[IntegerPart[b*2^k-b*j/c],
{j,0,IntegerPart[c*2^k]}];
points=Table[Disk[{x[[i+1]],y[[j+1]]},0.05],{j,0,IntegerPart[c*2^k]},{i,imin[[j+1]]+IntegerPart[a*2^k],imax[[j+1]]+IntegerPart[a*2^k]}];
Graphics[{RGBColor[0,0,1],GraphicsGroup[{triangle,points}]},GridLines→Automatic,
Axes→Automatic,AspectRatio→Automatic,
PlotRange→{{-a-1,b+1},{-1,c+1}}]
Neighbour:=Function[{i,j},nbs={};
If[i+1≤imax[[j+1]],
nbs=Append[nbs,{i+1,j}]];
If[imin[[j+1]]≤i-1,nbs=Append[nbs,{i-1,j}]];
```

```
If[(j<IntegerPart[c*2^k]) && (imin[[j+2]] ≤i)
&&(i≤imax[[j+2]]),bs=Append[nbs,{i,j+1}]];
If[(j>0) && (imin[[j]] ≤i) && (i≤ imax[[j]]),
nbs=Append[nbs,{i,j-1}]];
nbs[[RandomInteger[{1,Length[nbs]}]]]];
```

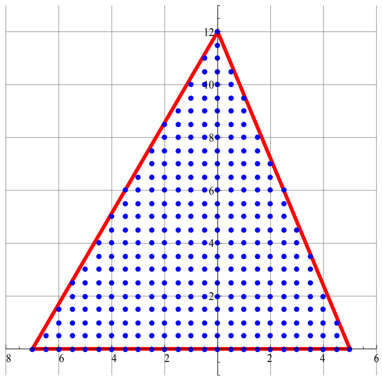

Fig. 1. The approximating domain D_1

Increasing the discretization level to k=4, we obtain more points-neighbors, as in Fig. 2 below.

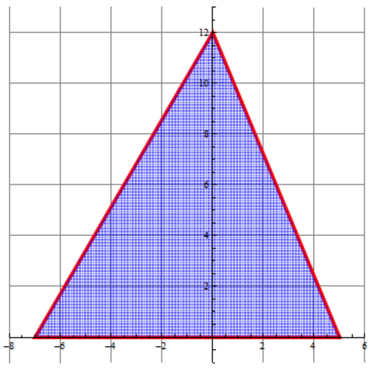

Fig. 2. The approximating domain D_4

For a given function f we will obtain a numerical approximation of the expected value $E^x f(B_t)$ with respect to a Brownian motion in a triangle starting at x of the value of the function f at the point B_t, value that correspondes to the solution of the Dirichlet Problem $u(x) = E^x f\left(B_{\tau_D}\right)$, in the given triangular region.

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